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Effective Lagrangians and Chiral Random Matrix Theory

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Abstract

Recently, sum rules were derived for the inverse eigenvalues of the Dirac operator. They were obtained in two different ways: i) starting from the low-energy effective Lagrangian and ii) starting from a random matrix theory with the symmetries of the Dirac operator. This suggests that the effective theory can be obtained directly from the random matrix theory. Previously, this was shown for three or more colors with fundamental fermions. In this paper we construct the effective theory from a random matrix theory for two colors in the fundamental representation and for an arbitrary number of colors in the adjoint representation. We construct a fermionic partition function for Majorana fermions in Euclidean space time. Their reality condition is formulated in terms of complex conjugation of the second kind.

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1 Introduction

Although it has been widely accepted that QCD is the correct theory for strong interactions, the nonlinearities in the interactions have made it very difficult to obtain accurate results that can be compared to experiment. In order to obtain rigorous results many researchers have studied a parameter range for which the theory simplifies but nevertheless contains the essential features of QCD. Well-known examples are e.g., the large N_c expansion [1, 2], the small volume expansion [3, 4], 2d theory [5], etc.. Such methods have provided us with numerous insights in the physics of the full theory justifying any new proposal in this direction.

Recently, Leutwyler and Smilga [6] proposed to focus on the quark mass dependence of the Euclidean QCD partition function close the the chiral limit (quark masses $m \ll \Lambda_{\rm QCD}$) in volumes with length scale L much larger than a typical hadronic length scale ($\Lambda_{\rm QCD}^{-1}$) but still much smaller than the pion Compton wave length ($\sim 1/\sqrt{m\Lambda_{QCD}}$). The first condition assures that only the low-lying excitations, in particular the Goldstone modes, contribute to the partition function, whereas the second condition allows us to ignore the kinetic terms in the Lagrangian. The mass dependence of the partition function is then given by the Lagrangian [6]

$$\mathcal{L} = mV\Sigma{\rm Re}({\rm TrU}), \tag{1.1}$$

where Σ is the vacuum expectation value of $\bar\psi\psi$ which is assumed to be nonzero, and the unitary matrix U parameterizes the Goldstone fields. The question that was asked in [6] is to what extent the knowledge of the finite volume partition function puts constraints of the spectrum of the Euclidean Dirac operator. By expanding both the QCD partition function and the partition function corresponding to (1.1) in powers of m, it was found, by equating the coefficients, that the inverse powers of the eigenvalues satisfy the Leutwyler-Smilga sum rules. Because of the Banks-Casher formula [7] the smallest nonzero eigenvalue is of order $1/\Lambda_{QCD}^3 L^4$, whereas in the absence of interactions, the smallest eigenvalue is of order $1/L$. The sum rules hold for eigenvalues well below the latter scale, which for sufficiently large volumes is well separated from the scale of the smallest eigenvalues.

In previous works [8, 9, 10, 11] we have investigated several questions related to the nature of these sum rules. In particular, it was argued that since the effective theory is solely based on the symmetries of QCD, the sum rules depend on the *symmetries* of the Dirac operator as well. So when one starts out with a theory with the same global symmetries as QCD but no other dynamical input one should arrive at exactly the same sum rules. We have constructed such theories: chiral random matrix theories, which apart from the chiral symmetry and possible anti-unitary symmetries also contain a remnant of the topological structure of QCD. In the framework of random matrix theory one has three different universality classes, those with real, complex or quaternion real matrix elements [12]. For $SU(2)$ with fundamental fermions the matrix elements are real, they are complex for more than two colors with fundamental fermions, whereas they are quaternion real for adjoint fermions. Indeed, it was shown that the sum rules that are obtained from the chiral random matrix theories [10] coincide with those obtained from the effective theory [6, 11]. It should be noted that the sum rules for $SU(2)$ were found only after the introduction of chiral random matrix theories [10]. The three classes of random matrix theories correspond to the three different schemes of chiral symmetry breaking, which were discussed before in the literature [13].

This suggests that it is possible to derive the finite volume partition function directly from the random matrix theory. In fact, this task has been performed for QCD with three or more colors [8]. However, for QCD with two colors or QCD with adjoint fermions, the situation is more complicated, and up to now such equivalence has not been proved. The main objective of this paper is to show that also in these two cases there is a one to one correspondence between chiral random matrix theory and the low energy finite volume partition function.

The main complication is the presence of anti-unitary symmetries in the QCD Lagrangian. For $N_c = 2$ this symmetry leads to a real Dirac operator and apart from a somewhat more complicated algebra it is straightforward to obtain the effective Lagrangian. To deal with adjoint fermions we have to face the well-known assertion that Majorana fermions do not exist in Euclidean space time [14]. However, this statement refers to the transformation properties of a Dirac spinor under Lorentz transformations. It does not exclude the possibility to write down a partition function for Majorana fermions

in terms of Grassmann integrals (see, for example, [15]). Starting from the observation that the adjoint Euclidean Dirac operator is anti-symmetric up to a charge conjugation matrix, we indeed succeeded to do this. In fact, using conjugation of the second kind [16, 17, 18], the Majorana condition in Euclidean space time is completely analogous to the one in Minkowski space time.

In a previous work [11], it was observed that the simplest sum rule for each of the three cases could be summarized by one formula involving the dimension of the Goldstone manifold of the theory. In order to show that this was no coincidence, we present a derivation that leads to this result naturally.

The structure of this paper is as follows. The symmetries of the Dirac operator for fundamental and adjoint fermions are discussed in sections 2 and 3, respectively. In section 4 we discuss the random matrix theory with these symmetries as input. The effective theories for the three different cases are derived in sections 5a, 5b and 5c. A general derivation of the simplest sum rule is given in section 6 and concluding remarks are made in section 7.

2 Symmetries of the Dirac operator for fundamental fermions

In this section we study the symmetries of the Euclidean Dirac operator[1]

$$i\gamma_\mu D_\mu = i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu, \tag{2.1}$$

for a fixed background field A_μ in the fundamental representation of $SU(N_c)$. For $N_c \geq 3$, the $N_c \times N_c$ matrix A_μ is complex valued, and the only symmetry of the Dirac operator is the chiral symmetry

$$\{i\gamma D, \gamma_5\} = 0. \tag{2.2}$$

[1]Our conventions are that the Euclidean gamma matrices are Hermitean and satisfy the anti-commutation relations $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$. We use a chiral representation in which γ_5 is diagonal.

Because of this the eigenvalues occur in pairs $\pm\lambda$ or are zero. The eigenfunctions are related by $\phi_\lambda = \gamma_5 \phi_{-\lambda}$. For $\lambda = 0$ we have the possibility that $\gamma_5 \phi_{\lambda=0} = \pm\phi_{\lambda=0}$, resulting in a fermionic zero mode that is either right handed or left handed. It is well known that this happens in the field of an instanton [19].

Because of the symmetry (2.2), the matrix representation of the Dirac operator simplifies in a chiral basis $\{\phi_{R\,k}, \phi_{L\,k},\}$ with $\gamma_5 \phi_{R\,k} = \phi_{R\,k}$ and $\gamma_5 \phi_{L\,k} = -\phi_{L\,k}$. The action becomes

$$\sum_{f,g=1}^{N_f} \sum_{kl} \left(\begin{array}{c} \chi_R^f \\ \chi_L^f \end{array} \right)_k^* \left(\begin{array}{cc} i\mathbf{m}_{fg}^* & D_{RL} \\ D_{LR} & i\mathbf{m}_{fg} \end{array} \right)_{kl} \left(\begin{array}{c} \chi_R^g \\ \chi_L^g \end{array} \right)_l , \tag{2.3}$$

where we have also included a mass matrix $\mathbf{m}^*(1+\gamma_5)/2 + \mathbf{m}(1-\gamma_5)/2$.

The matrix elements of D_{RL} given by

$$\int d^4x \phi_{R\,k}^* \, i\gamma D \, \phi_{L\,l} \tag{2.4}$$

are flavor diagonal. For a Hermitean Dirac operator we have $D_{LR} = D_{RL}^\dagger$. In the presence of fermionic zero modes the number of left handed and right handed states is not necessary equal. In general, the matrix D_{RL} is a rectangular one. Indeed, the total number of zero eigenvalues of the matrix in (2.3) in the chiral limit ($m \to 0$) is given by the absolute value of the difference in dimensionality of the left handed and the right handed Hilbert spaces.

For $N_c = 2$ the Euclidean Dirac operator is given by

$$i\gamma_\mu D_\mu = i\gamma_\mu \partial_\mu + \gamma_\mu A_\mu^a \frac{\tau_a}{2}, \tag{2.5}$$

where τ_k are the Pauli spin matrices ($\tau_1\tau_2 = i\tau_3$, etc.). In addition to the chiral symmetry (2.2), this Dirac operator possesses the anti-unitary symmetry:

$$[i\gamma D, C\tau_2 K] = 0 \quad \text{for} \quad N_c = 2. \tag{2.6}$$

Here, C is the charge conjugation matrix ($C = \gamma_2\gamma_4$ and $C^2 = -1$) and K is the complex conjugation operator. The anti-unitary operator $C\tau_2 K$ satisfies

$$(C\tau_2 K)^2 = 1 \tag{2.7}$$

From the analogy with the time reversal symmetry in quantum mechanics [20] it is clear that this condition allows us to choose a basis in which $i\gamma D$ is real. To show this, we

construct a basis $\{\psi_k\}$ that diagonalizes $C\tau_2 K$ with eigenvalues one,

$$C\tau_2\psi_k^* = \psi_k. \tag{2.8}$$

We start with an arbitrary basis vector ϕ_1, then $\psi_1 = \phi_1 + C\tau_2\phi_1^*$ satisfies (2.8). Next we choose ϕ_2 perpendicular to ψ_1. The second basis vector satisfying (2.8) is given by $\psi_2 = \phi_2 + C\tau_2\phi_2^*$, and it can be shown easily that it is orthogonal to ψ_1. The next basis vector is obtained from ϕ_3 perpendicular to both ψ_1 and ψ_2, etc..

By using (2.6) in the form

$$\tau_2 C\, i\gamma D C\, \tau_2 = -(i\gamma D)^*, \tag{2.9}$$

for which no analogy exists for $N_c \geq 3$, it follows immediately that the matrix elements (2.4) of the Dirac operator are real in this basis. Note that, because $[\gamma_5, C\tau_2 K] = 0$, the above construction holds true for a chiral basis. For $N_c = 2$, the fermionic part of the action is therefore given by (2.3), but with *real* valued matrices D_{RL} and D_{LR}. This makes it possible to rewrite (2.3) for $\mathbf{m} = \mathbf{0}$ as

$$\sum_{f=1}^{N_f} \sum_{kl} \begin{pmatrix} \chi_R^f \\ \chi_R^{f\,*} \end{pmatrix}_k D_{RL}^{kl} \begin{pmatrix} -\chi_L^{f\,*} \\ \chi_L^f \end{pmatrix}_l , \tag{2.10}$$

showing that the chiral symmetry group is enlarged to $U(2N_f)$. A mass matrix proportional to the identity breaks this symmetry to $Sp(2N_f)$. Below we will show that the same breaking pattern occurs by the formation of a quark condensate. It is important that the construction of the above basis relies on the *symmetries* of the Dirac operator only, and that the Dirac matrix becomes real for an arbitrary $SU(2)$ color gauge field.

3 Symmetries of the Dirac operator for Majorana fermions

The situation with Majorana fermions in the adjoint representation of the gauge group is much more complicated. The reality condition that defines Majorana fermions and their partition function in Minkowski space cannot be simply generalized to Euclidean space time, and in the literature one can find statements that Majorana fermions do not exist

in Euclidean space [14]. Below we will make this more explicit and provide a construction of a fermionic partition function that works as well in Euclidean space as in Minkowski space.

The Euclidean Dirac operator in the adjoint representation given by

$$i\gamma_\mu D_\mu = i\gamma_\mu(\partial_\mu \delta_{ab} + f_{abc} A^c_\mu), \tag{3.1}$$

where A^c_μ is an $SU(N_c)$ background field distributed according to the gluonic action. The essential difference from the Dirac operator in the fundamental representation is that the long derivative is anti-symmetric under transposition, which is also true in Minkowski space. Below we will exploit this by using that the full Dirac operator is anti-symmetric up to a constant matrix with determinant one.

Apart from the chiral symmetry,

$$\{i\gamma_\mu D_\mu, \gamma_5\} = 0, \tag{3.2}$$

the Dirac operator (3.1) also has an anti-unitary symmetry

$$[i\gamma D, CK] = 0. \tag{3.3}$$

First notice that, because

$$(CK)^2 = -1, \tag{3.4}$$

it is not possible to repeat the construction for $N_c = 2$ in the fundamental representation resulting in a real matrix. To see this, remember that the construction was based on the diagonalization of the anti-unitary symmetry operator. Now, let us assume that we can find an eigenvector $CK\phi = \lambda\phi$. Then,

$$(CK)^2\phi = CK\lambda\phi = \lambda^*\lambda\phi, \tag{3.5}$$

which in view of (3.4) leads to an obvious contradiction. As a corollary it follows that ϕ and $CK\phi$ are linearly independent.

To analyze the implications of the symmetry (3.3) in Euclidean space, we first address the issue of obtaining a fermionic action for Majorana fermions in Minkowski space. In this case the Dirac operator in the adjoint representation satisfies the commutation relation

$$[i\gamma_\mu D_\mu, \gamma_2 K] = 0, \tag{3.6}$$

and the anti-unitary charge conjugation operator

$$(\gamma_2 K)^2 = 1. \tag{3.7}$$

Therefore, the operator $\gamma_2 K$ can be diagonalized, and a superselection rule can be imposed that restricts the partition function to states with eigenvalue 1. Such states, called Majorana fermions, can be parameterized by

$$\psi^M = \left(\begin{array}{c} \chi_R \\ -\sigma_2 \chi_R^* \end{array} \right). \tag{3.8}$$

Since $\gamma_2 K$ also commutes with Lorentz transformations[2], ψ^M transforms as a Dirac spinor under the Lorentz group. This result is usually stated as follows [14, 21]: χ_L and $-\sigma_2\chi_R^*$ transform in the same way under Lorentz transformations. The Majorana Lagrangian is thus given by

$$\bar{\psi}^M i\gamma D \psi^M, \tag{3.9}$$

where $\bar{\psi}^M = \gamma_4 \psi^{M*}$. Consistent with (3.7), the Minkowski Dirac operator is real in the basis (3.8).

Let us pursue the construction of Majorana fermions and their action along a different route. The crucial observation is that the matrix $\gamma_2\gamma_4 i\gamma D$ is antisymmetric under transposition. This allows us to write down a fermionic action with half as many degrees of freedom

$$\det{}^{1/2} i\gamma D = \det{}^{1/2}\gamma_2\gamma_4 i\gamma D = \int \mathcal{D}\psi \exp[\psi\gamma_2\gamma_4 i\gamma D\psi]. \tag{3.10}$$

Lorentz invariance follows immediately form the fact that that $\gamma_2\gamma_4\psi$ and $\gamma_4\psi^*$ transform in the same way under Lorentz transformations. This can be made more explicit by introducing a conjugation operation such that

$$\gamma_4\psi^* = \gamma_2\gamma_4\psi, \tag{3.11}$$

which, of course, is precisely the Majorana condition. Before attacking the problem of 4 dimensional Euclidean Majorana fermions let us first consider the problem in 1+1 dimensional Minkowski space and two dimensional Euclidean space.

[2]In a chiral basis they are given by [14] $\psi_L \to \Lambda_L\psi_L$ and $\psi_R \to \Lambda_R\psi_R$ with $\Lambda_L = \exp(\frac{i}{2}\vec{\sigma}(\vec{\omega} - i\vec{\nu}))$ and $\Lambda_R = \sigma_2\Lambda_L^*\sigma_2$.

In 1+1 dimensional Minkowski space time, the Dirac operator in the adjoint representation, $i\gamma_\mu D_\mu$ with $\gamma-$matrices defined by $\{\gamma_\mu, \gamma_\nu\} = 2g_{\mu\nu}$ (We use the convention $\gamma_0 = \sigma_2,\ \gamma_1 = i\sigma_1$) has the anti-unitary symmetry

$$[i\gamma D, K] = 0. \tag{3.12}$$

Since K also commutes with Lorentz transformations[3] the Lagrangian for Majorana fermions, defined by the condition $K\psi^M = \psi^M$, is given by [22]

$$\mathcal{L}^M = \bar{\psi}^M i\gamma D\psi^M. \tag{3.13}$$

Here, $\bar{\psi}^M = \gamma_0\psi^{M\,*} = \gamma_0\psi^M$.

As in 3+1 dimensions, we could have followed an alternative route leading to the same Majorana action. The starting point is the observation that $i\sigma_2 i\gamma D$ is antisymmetric with respect to transposition which allows us to write the square root of the fermion determinant as a Grassmann integral with only half as many degrees of freedom:

$$\det{}^{1/2} i\gamma D = \det{}^{1/2} i\sigma_2 i\gamma D = \int \mathcal{D}\psi \exp[\psi i\sigma_2 i\gamma D\psi]. \tag{3.14}$$

This expression is Lorentz invariant. Indeed, ψ satisfies the Majorana condition.

Let us now proceed to Euclidean Majorana fermions in two dimensions. The gamma matrices are defined by $\{\gamma_\mu, \gamma_\nu\} = 2\delta_{\mu\nu}$, and we use the representation $\gamma_0 = \sigma_2,\ \gamma_1 = \sigma_1$. The Dirac operator satisfies the commutation relation

$$[i\gamma_\mu D_\mu, i\sigma_2 K] = 0. \tag{3.15}$$

Because $(i\sigma_2 K)^2 = -1$ we cannot impose a Majorana condition of the form $i\sigma_2 K\psi = \psi$. However, the matrix $i\sigma_2 i\gamma D$ is anti-symmetric with allows us to halve the number of fermionic degrees of freedom in the partition function

$$\det{}^{1/2} i\gamma D = \det{}^{1/2} i\sigma_2 i\gamma D = \int \mathcal{D}\psi \exp[\psi i\sigma_2 i\gamma D\psi]. \tag{3.16}$$

This action is invariant under Euclidean Lorentz transformation, i.e. under $\psi \to \exp(i\sigma_3\phi)\psi$. This can be made more explicit by introducing a conjugation operator such that

$$\psi^* = -i\sigma_2\psi. \tag{3.17}$$

[3]For our representation of the gamma matrices, Lorentz transformations of spinors are given by $\psi \to \exp(-\omega\sigma_3)\psi$.

In components, the equations read

$$\psi_1^* = -\psi_2, \qquad \psi_2^* = \psi_1, \tag{3.18}$$

which forces us to impose the consistency condition $\psi_k^{**} = -\psi_k$. This conjugation, called conjugation of the second kind, is well-known in the mathematical literature on Grassmann variables [16], and has been used extensively in the supersymmetric formulation of random matrix theories [17, 18]. The Majorana constraint (3.17)

$$i\sigma_2 K \begin{pmatrix} \psi_1 \\ \psi_2^* \end{pmatrix} = \begin{pmatrix} \psi_1 \\ \psi_2^* \end{pmatrix}, \tag{3.19}$$

makes the symmetry (3.15) manifest at the level of the partition function.

Let us now return to 4 Euclidean dimensions. The strategy should be clear: we construct an anti-symmetric operator that differs from the Euclidean Dirac operator only by a factor with unit determinant. As can be seen from the following theorem, it is not an accident that this works.

Theorem. Consider a Hermitean operator H such that $[H, AK] = 0$ and $A^\dagger A = AA^\dagger = 1$. If $(AK)^2 = -K^2$ then $(HA)^T = -HA$. If, moreover A or iA are orthogonal then also $(AH)^T = -AH$.

The proof of this theorem is immediate. It can be applied to the Dirac operator in Euclidean space time. However, since going from Euclidean to Minkowski space just amounts to multiplying the spacial gamma matrices by a factor i, in both cases the Dirac operator behaves the same under transposition.

Using this theorem, one concludes from (3.3) that $Ci\gamma D$ is an antisymmetric matrix, which allows us to write down a fermionic partition function with only half as many degrees of freedom

$$\det{}^{1/2} i\gamma D = \det{}^{1/2} Ci\gamma D = \int \mathcal{D}\psi \exp[\psi Ci\gamma D\psi]. \tag{3.20}$$

This construction also works in the presence of a mass term. It is straightforward to verify that (3.20) is invariant under Euclidean Lorentz transformations. As before, this can be made more explicit by choosing the components of ψ such that they satisfy the conjugation equation

$$C\psi^* = \pm\psi. \tag{3.21}$$

A solution with eigenvalue $+1$ can be parametrized by

$$\psi = \begin{pmatrix} \psi_1 \\ \psi_1^* \\ \psi_2 \\ -\psi_2^* \end{pmatrix}, \tag{3.22}$$

which can be viewed as a Majorana constraint on the fermionic integration variables. As was the case for Euclidean fermions in 2 dimensions, we have to impose conjugation of the *second* kind on the Grassmann variables (otherwise the equation $CK\psi = \pm\psi$ does not have solutions, see eq. (3.5)).

To display the matrix structure of the Dirac operator we express the field in the Lagrangian (3.20) in terms of a complete set with Grassmannian coefficients. An expansion consistent with the Majorana constraint (3.21) is given by (because $(CK)^2 = -1$, the c-number functions ϕ and $C\phi^*$ are linearly independent)

$$\psi_R = \sum_k \phi_{R\,k}\chi_{R\,k} + C\phi^*_{R\,k}\chi^*_{R\,k} \tag{3.23}$$

for right handed fermions and,

$$\psi_L = \sum_k \phi_{L\,k}\chi_{L\,k} + C\phi^*_{L\,k}\chi^*_{L\,k} \tag{3.24}$$

for left handed fermions. In this basis the fermionic piece of the action is given by

$$\begin{pmatrix} \chi_R^f \\ \chi_R^{f\,*} \\ \chi_L^f \\ \chi_L^{f\,*} \end{pmatrix}_k^* \begin{pmatrix} \mathbf{m}^*_{fg} & D_{RL} \\ D_{LR} & \mathbf{m}_{fg} \end{pmatrix}_{kl} \begin{pmatrix} \chi_R^g \\ \chi_R^{g\,*} \\ \chi_L^g \\ \chi_L^{g\,*} \end{pmatrix}_l, \tag{3.25}$$

where each 2×2 block of D_{RL} is given by

$$(D_{RL})_{kl} = \begin{pmatrix} \int \phi^*_{R\,k}\, i\gamma D\, \phi_{L\,l} & \int \phi^*_{R\,k}\, i\gamma D\, C\phi^*_{L\,l} \\ \int \phi_{R\,k}\, C^T i\gamma D\, \phi_{L\,l} & \int \phi_{R\,k}\, C^T i\gamma D\, C\phi^*_{L\,l} \end{pmatrix}, \tag{3.26}$$

and $D_{LR} = D_{RL}^\dagger$. In (3.25) we have included the flavor indices. Using that

$$C^T i\gamma DC = (i\gamma D)^*, \tag{3.27}$$

$$C^T i\gamma D = -(i\gamma D)^* C, \tag{3.28}$$

we find that each 2×2 block in D_{RL} is quaternion real, i.e. of the form $a_0 + ia_k\sigma_k$ with a_μ real. The mass matrix is proportional to the unit quaternion but is generally not diagonal

in the flavor indices. Since the product of Grassmann variables that multiplies the mass matrix is symmetric in the flavor indices, the mass matrix can be taken symmetric as well.

Because the Grassmannian vectors on the left and on the right in (3.25) do not transform independently under transformations in flavor space, the chiral symmetry for $\mathbf{m} = 0$ is reduced to $U(N_f)$. For a nonzero mass matrix that is a multiple of the identity, only an invariance under the $O(N_f)$ subgroup remains. Below we will see that the same breaking is achieved by the formation of a nonzero chiral condensate.

The eigenvalues of the Dirac operator (3.25) are doubly degenerate and, for zero mass, occur in pairs $\pm\lambda$ or are zero. The number of zero eigenvalues is equal to *twice* the absolute value of the difference of the dimensionality of the right handed and the left handed space.

4 Chiral random matrix theories

The matrix elements of the Dirac operator fluctuate over the ensemble of gauge fields. In this section we introduce a model partition function with the *symmetries* of the Dirac operator but with matrix elements fluctuating according to independent gaussian distributions. As discussed in the introduction, we expect that certain low energy quantities, such as the fluctuations of the eigenvalues on a microscopic scale, do not depend on the dynamics of QCD interaction and can be calculated in this simplified model.

The QCD partition function can be written as

$$Z_{QCD} = \sum_{\bar\nu} e^{i\bar\nu\theta} Z_{QCD}(\bar\nu), \tag{4.1}$$

where the partition function in the sector with $\bar\nu$ fermionic zero modes is defined by

$$Z_{QCD}(\bar\nu) = \langle \prod_{f=1}^{N_f} m_f^{\bar\nu} \prod_{\lambda_n>0} (\lambda_n^2 + |m_f|^2) \rangle_A. \tag{4.2}$$

The eigenvalues of the mass matrix are denoted by m_f, and the average goes over all gauge fields with topological charge ν weighted according to the gluonic action. For fundamental

fermions $\bar{\nu} = \nu$, but for adjoint fermions, with each of the doubly degenerate eigenvalues included only once in the fermion determinant, we have $\bar{\nu} = N_c \nu$ (see [6]).

In sectors with a nonzero total topological charge, the number of right handed modes in (2.3) and (3.25) differs from the number of left handed modes. The random matrix partition function with n and $n + \bar{\nu}$ of such modes, respectively, and N_f flavors is defined by

$$Z_\beta(\bar{\nu}, N_f) = \int \mathcal{D}T \prod_{f=1}^{N_f} \det \begin{pmatrix} m_f^* & iT \\ iT^\dagger & m_f \end{pmatrix} \exp[-\frac{n\beta\Sigma^2}{2} \mathrm{Tr} T T^\dagger], \tag{4.3}$$

where T is an $n \times (n + \bar{\nu})$ matrix. The integration over T is according to the Haar measure. The matrix elements of T are real for $\beta = 1$ corresponding to QCD with $N_c = 2$ in the fundamental representation. They are complex for $N_c \geq 3$ in the fundamental representation ($\beta = 2$). For gauge fields in the adjoint representation the matrix elements of the Dirac operator are quaternion real (see (3.27)), and the matrix elements T_{ij} in (4.3) are chosen quaternion real as well ($\beta = 4$). If we write the quaternions in terms of 2×2 matrices, the matrix T is a $2n \times 2(n + \bar{\nu})$ matrix. Of course, the masses m_f are multiplied by the quaternion unit matrix. The determinant in (4.3) for $\beta = 4$ is the so called Qdet and the trace is the QTr. It can be shown that for a quaternion real matrix A that $\mathrm{Qdet}^2 A = \det A$ [30]. In a 2×2 matrix representation of the quaternions the QTr is just one half the ordinary trace. Note that C times the unit in the flavor indices multiplied by the matrix in (3.26) is antisymmetric which makes the square root of its determinant is well defined.

In analogy with the classical random matrix ensembles, these ensembles will be called, the chiral orthogonal ensemble (chGOE), the chiral unitary ensemble (chGUE) and the chiral symplectic ensemble (chGSE), for $\beta = 1$, 2, and 4, respectively.

We will identify the total number of modes, $2n$, (we always have $\bar{\nu} \ll n$) with the volume of space time. This corresponds to choosing units in which the density of the low-lying modes is equal to one. Below we will see that the parameter Σ can be identified as the chiral condensate.

The matrix ensembles described by the partition function (4.3) are equivalent to what is known in the random matrix literature as the Laguerre ensembles. They first were introduced by Fox and Kahn [23]. The simplest case $\beta = 2$ was analyzed in [24]. An analysis

of the $\beta = 1$ case in the context of the microscopic spectral density of the Dirac operator was given in [10]. An analysis which also includes many other correlation functions for $\beta = 1$ and $\beta = 4$ was performed in a series of papers by Nagao and coworkers and Forrester [25, 26]. Other results for the chiral random matrix ensembles have been obtained in terms of a supersymmetric formulation [27] The recent work on this subject is based on results by Mehta and Mahoux [28] who introduced the skew orthogonal polynomials originally invented by Dyson [29].

The fermion determinant in (4.1) can be written as a Grassmann integral

$$Z_\beta(\bar{\nu}, N_f) = \int \mathcal{D}T\mathcal{D}\psi^*\mathcal{D}\psi \exp[-i \sum_{f,g=1}^{N_f} \psi_i^{f\,*} \begin{pmatrix} i\mathbf{m}_{fg}^* & T \\ T^\dagger & i\mathbf{m}_{fg} \end{pmatrix}_{ik} \psi_k^g - \frac{n\beta\Sigma^2}{2}\mathrm{Tr}TT^\dagger]. \quad (4.4)$$

For adjoint fermions ($\beta = 4$) the matrix elements of T are quaternion real, and each component ψ_i^f is a vector of length four given by (see (3.26))

$$\psi_i^f = \begin{pmatrix} \chi_{R\,i}^f \\ \chi_{R\,i}^{*\,f} \\ \chi_{L\,i}^f \\ \chi_{L\,i}^{*\,f} \end{pmatrix}. \quad (4.5)$$

They satisfy a reality condition similar to the Majorana constraint. In order to assure ourselves of a positive definite fermion determinant for $\nu = 0$ we have included a factor i in front of the fermionic action. For $\beta = 1$ and $\beta = 2$ the components of ψ and ψ^* are independent integration variables.

5 Effective theory

In this section we derive the effective theory corresponding to the random matrix partition function (4.4). We proceed by averaging over the matrix T resulting in a four-fermion interaction which can be made gaussian at the expense of a new bosonic integration variable. After performing the Grassmann integrals the resulting theory is amenable to a saddle point approximation in which the integrals over the soft modes are kept and the integrals over the hard modes are done to gaussian order.

For completeness we start the discussion with the simplest case $\beta = 2$ (section 5a)

which was already analyzed in ref. [10]. In sections 5b and 5c we discuss the cases $\beta = 1$ and $\beta = 4$, respectively.

5.1 Effective theory for $\beta = 2$

After averaging over the matrix elements of the Dirac operator the partition function becomes

$$Z_2(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \exp[-\frac{2}{n\Sigma^2\beta}\psi^{f\,*}_{L\,k}\psi^{f}_{R\,i}\psi^{g\,*}_{R\,i}\psi^{g}_{L\,k} + (\mathbf{m}^*_{fg}\psi^{f\,*}_{R\,i}\psi^{g}_{R\,i} + \mathbf{m}_{fg}\psi^{f\,*}_{L\,k}\psi^{g}_{L\,k})]. \tag{5.1}$$

Here and below we have used the $\sim$ sign in order to indicate that constant factors have been absorbed in the normalization of the partition function. The term of fourth order in the Grassmann variables can be rewritten as the difference of two squares,

$$\psi^{f\,*}_{L\,k}\psi^{g}_{L\,k}\psi^{g\,*}_{R\,i}\psi^{f}_{R\,i} = \frac{1}{4}(\psi^{f\,*}_{L\,k}\psi^{g}_{L\,k} + \psi^{g\,*}_{R\,i}\psi^{f}_{R\,i})^2 - \frac{1}{4}(\psi^{f\,*}_{L\,k}\psi^{g}_{L\,k} - \psi^{g\,*}_{R\,i}\psi^{f}_{R\,i})^2 \tag{5.2}$$

Using the Hubbard-Stratonovitch transformation, each of the squares can be linearized by introducing an additional Gaussian integral

$$\exp(-AQ^2) \sim \int d\sigma \exp(-\frac{\sigma^2}{4A} - iQ\sigma). \tag{5.3}$$

This results in the partition function

$$\begin{aligned} Z_2(\bar{\nu}, N_f) \;&\sim\; \int \mathcal{D}\sigma_1 \mathcal{D}\sigma_2 \mathcal{D}\psi \mathcal{D}\psi^* \exp[-\frac{n\Sigma^2\beta}{2}\mathrm{Tr}(\sigma_1\sigma_1^T + \sigma_2\sigma_2^T) \\ &+\; \psi^{f\,*}_{R\,i}\psi^{g}_{R\,i}(\sigma_1^{fg} - i\sigma_2^{fg} + \mathbf{m}^*_{fg}) + \psi^{f\,*}_{L\,k}\psi^{g}_{L\,k}(\sigma_1^{fg} + i\sigma_2^{fg} + \mathbf{m}_{fg})], \end{aligned} \tag{5.4}$$

where σ_1 and σ_2 are arbitrary real $N_f \times N_f$ matrices. The integration over the Grassmann variables yields

$$Z_2(\bar{\nu}, N_f) \sim \int \mathcal{D}A \mathrm{det}^{n+\nu}(A^\dagger + \mathbf{m}^*)\mathrm{det}^n(A + \mathbf{m}) \exp[-\frac{n\Sigma^2\beta}{2}\mathrm{Tr}AA^\dagger], \tag{5.5}$$

where A is an arbitrary complex matrix. It can be diagonalized according to

$$A = U\Lambda V^{-1}. \tag{5.6}$$

In order to have the same number of degrees of freedom on both sides of the equation we chose $U \in U(N_f)$ and $V \in U(N_f)/(U(1))^{N_f}$. All matrix elements of the diagonal matrix

Λ are real non-negative. For $n \to \infty$ and $\|\mathbf{m}\|\Sigma \ll 1$, the integral over the eigenvalues can be performed by a saddle point approximation at $\mathbf{m} = 0$. The solutions of the saddle point equation for Λ are given by

$$\Lambda_k = \pm \frac{1}{\Sigma}, \tag{5.7}$$

but only the solution with all signs positive is inside the integration manifold. At this point, the integral only depends on the combination UV^{-1} which allows us to absorb V in the integration over U and perform the V integration. For small masses, $m_f \Sigma \ll 1$ (the eigenvalues of the mass matrix are denoted by m_f), the integral over U is soft and cannot be done by a saddle point method. However, in this limit we can expand the determinant to first order in $\mathbf{m}$ which leads to the partition function

$$Z_2(\bar{\nu}, N_f) \sim \int d\theta \exp(i\bar{\nu}\theta) \int_{U \in SU(N_f)} \mathcal{D}U \exp[n\Sigma \mathrm{Tr}(\mathbf{m}^* U e^{i\theta/N_f} + \mathbf{m} U^{-1} e^{-i\theta/N_f})], \tag{5.8}$$

where we have split the integration over U in a $U(1)$ integral over θ and an integral over $SU(N_f)$ ($U \to U \exp[i\theta/N_f]$). For a diagonal mass matrix the condensate is given by

$$\langle \bar{q}_f q_f \rangle = \frac{1}{2n} \partial_{m_f} \log Z, \tag{5.9}$$

where the differentiation is with respect to *one* of the quark masses. The right hand side should be evaluated for $m_f n \Sigma \gg 1$, which allows us to use a saddle point integration for U with the result that

$$\langle \bar{q}_f q_f \rangle = \Sigma \cos(\frac{\theta}{N_f}). \tag{5.10}$$

This completes the calculation of the partition function that, with the identification of $2n$ as the volume of space-time, was the starting point of ref. [6].

5.2 Effective theory for $\beta = 1$

For $\beta = 1$, the overlap matrix T is real. After averaging over T the partition function is given by

$$\begin{aligned} Z_1(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \qquad & \exp\Big[-\frac{1}{2n\Sigma^2\beta} \left[\begin{pmatrix} \psi_{R\,i} \\ \psi^*_{R\,i} \end{pmatrix}_f I_{ff'} \begin{pmatrix} \psi_{L\,k} \\ \psi^*_{L\,k} \end{pmatrix}_{f'} \right]^2 \\ & + \ \mathbf{m}^*_{fg} \psi^{f\,*}_{R\,i} \psi^g_{R\,i} + \mathbf{m}_{fg} \psi^{f\,*}_{L\,k} \psi^g_{L\,k} \Big]. \end{aligned} \tag{5.11}$$

The introduction of anti-symmetric unit matrix

$$I = \begin{pmatrix} 0 & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix}, \tag{5.12}$$

allows us to rearrange the fermions in multiplets of length $2N_f$, in agreement with the well-known result [13] that for two colors the the chiral symmetry group is enlarged from $U(N_f) \times U(N_f)$ to $U(2N_f)$. Since baryons consist of two quarks for $N_c = 2$, it is not surprising that the four-fermion interaction contains both mesonic and di-quark bilinears. It can be rewritten as the difference of

$$\frac{1}{4}\left[\begin{pmatrix} \psi_{R\,i} \\ \psi^*_{R\,i} \end{pmatrix}_f \begin{pmatrix} \psi_{R\,i} \\ \psi^*_{R\,i} \end{pmatrix}_g + I_{gg'} \begin{pmatrix} \psi_{L\,k} \\ \psi^*_{L\,k} \end{pmatrix}_{g'} \begin{pmatrix} \psi_{L\,k} \\ \psi^*_{L\,k} \end{pmatrix}_{f'} I_{f'f} \right]^2 \tag{5.13}$$

and a similar expression with the plus sign exchanged by a minus sign. Each of the fermionic bilinears is anti-symmetric in the flavor indices. Using the Hubbard-Stratonovitch transformation (5.3), the squares can be linearized with the help of an anti-symmetric matrix which yields the following partition function

$$\begin{aligned} Z_1(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^*\mathcal{D}\psi \exp\big[-2n\Sigma^2\beta \mathrm{Tr}(AA^\dagger) &+ \begin{pmatrix} \psi_R \\ \psi^*_R \end{pmatrix}_f \Big(A^\dagger + \frac{1}{2}\mathcal{M}^*\Big)_{fg} \begin{pmatrix} \psi_R \\ \psi^*_R \end{pmatrix}_g \\ &+ \begin{pmatrix} \psi_L \\ \psi^*_L \end{pmatrix}_f \Big(IAI^T + \frac{1}{2}\mathcal{M}\Big)_{fg} \begin{pmatrix} \psi_L \\ \psi^*_L \end{pmatrix}_g \big]. \end{aligned} \tag{5.14}$$

The mass matrix is defined by

$$\mathcal{M} = \begin{pmatrix} 0 & -\mathbf{m} \\ \mathbf{m} & 0 \end{pmatrix}, \tag{5.15}$$

and A is a general antisymmetric complex $2N_f \times 2N_f$ matrix. Carrying out the fermion integrals and rescaling A by a factor two, the partition function can be compactly written as

$$Z_1(\bar{\nu}, N_f) \sim \int \mathcal{D}A \mathrm{Pf}^{n+\nu}(A^\dagger + \mathcal{M}^*)\mathrm{Pf}^n(A + \mathcal{M}) \exp[-\frac{n\beta\Sigma^2}{2}\mathrm{Tr}AA^\dagger], \tag{5.16}$$

where Pf denotes the Pfaffian of the matrix (For an even dimensional anti-symmetric matrix, the Pfaffian is equal to the square root of its determinant with a definite choice for its phase [31, 11]). The complex antisymmetric matrix A can be brought into a standard form as

$$A = U\Lambda U^T, \tag{5.17}$$

where U is a unitary matrix and Λ a real antisymmetric matrix with $\Lambda_{k,k+1} = -\Lambda_{k+1,k} = \lambda_k$, $k = 1, \cdots, 2N_f - 1$ and all other matrix elements zero. By redefining U we can always choose all $\lambda_k \geq 0$. In (5.16) we use Λ and U as new integration variables. Note that $U \in U(2N_f)/(Sp(2))^{N_f}$ so that the total number of degrees of freedom on both sides of (5.17) is the same. We are interested in the limit $n \to \infty$ and $\|\mathcal{M}\|\Sigma \ll 1$. Then the Λ integration can be performed by a saddle point integration at $\mathcal{M} = 0$, whereas the remaining integrals have to be performed exactly for the actual value of the mass. At the saddle point inside the integration manifold, i.e. $\lambda_k = 1/\Sigma$, the integrand does not depend on U for $\mathcal{M} = 0$. For $\mathcal{M} \neq 0$ the U dependence is in the form UIU^T, where I is the antisymmetric unit matrix. The integration over the soft modes U is thus parametrized by the coset $U(2N_f)/Sp(2N_f)$. For $\|\mathcal{M}\|\Sigma \ll 1$ the exponentiated determinants can be expanded to first order in $\mathcal{M}$ resulting in the low energy finite volume partition function

$$Z_1(\bar{\nu}, N_f) \sim \int d\theta \exp(i\bar{\nu}\theta) \int_{U \in SU(2N_f)/Sp(2N_f)} \exp[n\Sigma \mathrm{Re}(e^{i\theta/N_f} \mathrm{Tr} UIU^T \mathcal{M})], \tag{5.18}$$

which was used as starting point for the calculation of the Leutwyler-Smilga sum rules in [11] (with the identification of $2n$ as the volume of space-time). The phase of $\det U$ has been isolated by the substitution $U \to U \exp[i\theta/2N_f]$. Note that in this way the phase of the Pfaffian in (5.16) covers the full complex unit circle for $\theta \in [0, 2\pi]$. The integration over the stability subgroup only modifies the partition function by a constant. Therefore, the integration in (5.18) can be extended to $SU(N_f)$, which facilitates further evaluation of the partition function [11].

Also in this case the condensate is given by the logarithmic derivative in (5.9) evaluated for $m_f n\Sigma \gg 1$ (with m_f an eigenvalue of $\mathbf{m}$). In this limit the U integral can be performed by a saddle point method. The saddle point is at $UIU^T = 1$ resulting in the identification

$$\langle \bar{q}_f q_f \rangle = \Sigma \cos(\frac{\theta}{N_f}). \tag{5.19}$$

5.3 Effective theory for $\beta = 4$

In this case the overlap matrix elements are quaternion real

$$T_{ik} = \sum a_{ik}^{\mu} i\sigma_{\mu}^{+}, \tag{5.20}$$

where $\sigma_\mu^+ = (-i, \vec{\sigma})_\mu$ and the a_{ik}^μ are real. In terms of the a_μ variables the partition function reads

$$Z_4(\bar{\nu}, N_f) \sim \int \mathcal{D}a^\mu \mathcal{D}\psi^* \mathcal{D}\psi \qquad \exp\left[-\frac{n\beta\Sigma^2}{2} \mathrm{Tr} a_\mu a_\mu^T + \psi_{Ri}^{f\,*} a_{ik}^\mu \sigma_\mu^+ \psi_{Lk}^f - \psi_{Lk}^{f\,*} a_{ik}^\mu (\sigma_\mu^+)^\dagger \psi_{Ri}^f \right. \\ \left. + \ \mathbf{m}_{fg}^* \psi_{Ri}^{f\,*} \psi_{Ri}^g + \mathbf{m}_{fg} \psi_{Lk}^{f\,*} \psi_{Lk}^g \right]. \tag{5.21}$$

The components of the fermionic variables are 2 component vectors defined by

$$\psi_R = \begin{pmatrix} \chi_R \\ \chi_R^* \end{pmatrix}, \qquad \psi_L = \begin{pmatrix} \chi_L \\ \chi_L^* \end{pmatrix}. \tag{5.22}$$

By using the identity $((\sigma_\mu^+)^\dagger)^T = -\sigma_2 \sigma_\mu^+ \sigma_2$ we find that the two fermionic terms in (5.21) are identical. Averaging over the overlap matrix elements leads to

$$Z_4(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \exp\left[\frac{2}{n\beta\Sigma^2} \psi_{Ri}^{f\,*} \sigma_\mu^+ \psi_{Lk}^f \psi_{Ri}^{g\,*} \sigma_\mu^+ \psi_{Lk}^g + \mathbf{m}_{fg}^* \psi_{Ri}^{f\,*} \psi_{Ri}^g + \mathbf{m}_{fg} \psi_{Lk}^{f\,*} \psi_{Lk}^g\right]. \tag{5.23}$$

Using the Fierz identity

$$\sum_\mu \sigma_\mu^{+\,\alpha\beta} \sigma_\mu^{+\,\gamma\delta} = 2(\delta_{\alpha\delta}\delta_{\gamma\beta} - \delta_{\alpha\beta}\delta_{\gamma\delta}) \tag{5.24}$$

and the representation (5.22) of the spinors, the partition function can be simplified to

$$Z_4(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \qquad \exp\left[\frac{4}{n\beta\Sigma^2} (\chi_{Ri}^{f\,*} \chi_{Ri}^g + \chi_{Ri}^{g\,*} \chi_{Ri}^f)(\chi_{Lk}^{f\,*} \chi_{Lk}^g + \chi_{Lk}^{g\,*} \chi_{Lk}^f) \right. \\ \left. + \ \mathbf{m}_{fg}^* \psi_{Ri}^{f\,*} \psi_{Ri}^g + \mathbf{m}_{fg} \psi_{Lk}^{f\,*} \psi_{Lk}^g\right]. \tag{5.25}$$

This four fermion interaction can be written as the difference of two squares which can be linearized with the help of the Hubbard-Stratonovitch transformation (5.3) by introducing the bosonic variables σ_1 and σ_2. Both σ_1 and σ_2 are symmetric real valued $N_f \times N_f$ matrices. In terms of the new variables the partition function reads

$$Z_4(\bar{\nu}, N_f) \sim \int \mathcal{D}\psi^* \mathcal{D}\psi \mathcal{D}\sigma_1 \mathcal{D}\sigma_2 \exp\left[-\frac{n\beta\Sigma^2}{4} \mathrm{Tr}(\sigma_1^2 + \sigma_2^2)\right] \\ \times \ \exp\left[2\chi_{Ri}^{f\,*} \chi_{Ri}^g (\sigma_1 - i\sigma_2)_{fg} + 2\chi_{Lk}^{f\,*} \chi_{Lk}^g (\sigma_1 + i\sigma_2)_{fg} + 2(\mathbf{m}_{fg}^* \chi_{Ri}^{f\,*} \chi_{Ri}^g + \mathbf{m}_{fg} \chi_{Lk}^{f\,*} \chi_{Lk}^g)\right]. \tag{5.26}$$

The Grassmann integrals can be performed trivially resulting in the partition function

$$Z_4(\bar{\nu}, N_f) \sim \int \mathcal{D}A \det{}^{n+\nu}(A^\dagger + \mathbf{m}^*) \det{}^n (A + \mathbf{m}) \exp\left[-\frac{n\beta\Sigma^2}{4} \mathrm{Tr} A A^\dagger\right], \tag{5.27}$$

where A is a complex symmetric matrix. Such a real matrix can be diagonalized by a unitary matrix as follows

$$A = U\Lambda U^T \tag{5.28}$$

with Λ a real nonnegative diagonal matrix. As before we use Λ and U as new integration variables. The mass plays the role of a small symmetry breaking term ($\|\mathbf{m}\|\Sigma \ll 1$) so that the integration over the hard modes, the Λ, can be performed by a saddle point approximation at $\mathbf{m} = 0$. The saddle points are given by

$$\lambda_k = \pm\frac{1}{\Sigma}, \tag{5.29}$$

but only the saddle point with all signs positive is inside the integration manifold. The integral over the soft modes has to be taken into account exactly, but the exponentiated determinant can be expanded to first order in $\mathbf{m}$. Writing U as the product of a phase and a special unitary transformation ($U \to \exp(i\theta/2N_f)U$) we arrive at the partition function

$$Z_4(\bar\nu, N_f) \sim \int d\theta e^{i\bar\nu\theta} \int_{U\in SU(N_f)/O(N_f)} \mathcal{D}U \exp[2n\Sigma \mathrm{Re}(e^{i\theta/N_f}\mathrm{Tr} UU^T\mathbf{m})]. \tag{5.30}$$

Because the integrand depends only on U^TU, the integration over U is effectively over the coset $SU(N_f)/O(N_f)$, where the volume of the stability subgroup can be absorbed in the normalization. As before, the condensate is given by the logarithmic derivative of the partition function for $m_f n\Sigma \gg 1$ (The eigenvalues of $\mathbf{m}$ are denoted by m_f). The integral can than be performed by a saddle point method resulting in the identification

$$\langle \bar q_f q_f\rangle = \Sigma\cos(\frac{\theta}{N_f}). \tag{5.31}$$

With the identification of $2n$ as the volume of space-time, the partition function (5.30) was the starting point for the derivation of the Leutwyler-Smilga sum rules in [6, 11].

6 Sum rules for the eigenvalues of the Dirac operator

As already mentioned in the introduction, the sum of inverse powers of the eigenvalues of the Dirac operator satisfies sum rules. Since they have been derived both from effective

field theory and random matrix theory, we want to restrict ourselves to one feature that was not clarified in earlier work.

To obtain sum rules we expand both the expectation value of the fermion determinant and the finite volume partition function in powers of m^2 (We consider the case of a diagonal mass matrix with all masses equal to m). For the fermion determinant we obtain

$$\frac{Z(m)}{Z(0)} = 1 + m^2 N_f \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2}, \tag{6.1}$$

where, for simplicity, we consider the case with all masses equal. Equating the coefficients of m^2 in (6.1) and of the expansion of the finite volume partition function (see below), leads to sum rules that can be summarized into one formula [11] valid for each of the three cases discussed in section 5:

$$\frac{1}{V^2} \sum_{\lambda_n > 0} \frac{1}{\lambda_n^2} = \frac{\Sigma^2}{4(|\nu| + (\text{dim(coset)} + 1)/N_f)}. \tag{6.2}$$

The volume of space-time is denoted by V. In this section we give a derivation for $\nu = 0$ showing that this unifying feature was no accident.

The coefficient of m^2 of the ratio $Z(m)/Z(0)$ for $\nu = 0$ of finite volume partition functions given in (5.8), (5.18) and (5.30) involves the calculation of integrals of the form

$$\zeta(A) = \int_{V \in G/H} dV \, \text{Tr}(VA) \text{Tr}^*(VA), \tag{6.3}$$

For $\nu = 0$ the $U(1)$ integral over θ can be absorbed in the integration over the coset G/H, which now becomes $U(2N_f)/Sp(2N_f)$, $U(N_f)$ and $U(N_f)/SO(N_f)$ for $\beta = 1$, 2 and 4, respectively. The matrix A is the mass matrix which is real anti-symmetric, complex and real symmetric in this order. Using the invariance of the measure it follows that in all three cases $\zeta(A) \sim \text{Tr} A A^\dagger$. For a mass matrix with equal masses in the standard form, $A \sim \mathbf{1}$ for $\beta = 2$ and $\beta = 4$ and proportional to the antisymmetric unit matrix I for $\beta = 1$.

To proceed let us introduce generators t_k of the cosets [11]. They satisfy the orthogonality relations

$$\text{Tr}\, t_k t_l = \frac{1}{2} \delta_{kl}, \tag{6.4}$$

and are real anti-symmetric, Hermitean and real symmetric, for $\beta = 1$, $\beta = 2$ and $\beta = 4$, respectively. The total number of generators is denoted by M and the dimension of

the matrices U and t_k is denoted by d. The generators will be chosen such that t_1 is proportional to the mass matrix. Because of the invariance discussed in the previous paragraph and the normalization (6.4), we find that

$$\zeta(t_1) = \zeta(t_2) = \cdots = \zeta(t_M), \tag{6.5}$$

allowing us to rewrite the integral as

$$\zeta(t_1) = \frac{1}{M} \sum_{k=1}^{M} \int_{V \in G/H} dV \, \mathrm{Tr}(V t_k) \mathrm{Tr}^*(V t_k). \tag{6.6}$$

By expanding U in its generators ($U = \sum_k u_k t_k$) and using the orthogonality relation (6.4) and the unitarity of U, the integral becomes trivial. The result is given by

$$\zeta(t_1) = \frac{d}{2M} \mathrm{vol}(G/H). \tag{6.7}$$

The volume of the coset cancels in the ratio $Z(m)/Z(0)$ of the partition functions. Using that the dimensionality $d = N_f$ for $\beta = 2$ and $\beta = 4$ and $d = 2N_f$ for $\beta = 1$, we find that

$$\left. \frac{Z(m)}{Z(0)} \right|_{\nu=0} = 1 + \frac{1}{4} m^2 V^2 \Sigma^2 \frac{N_f}{2M} 2N_f. \tag{6.8}$$

The extra factor $2N_f$ appeared because of the difference in normalization between the unit matrix and the generators (6.4). As already discussed in section 4, we have identified the total number of zero modes, $2n$, with the volume V of space-time.

In (6.2) the dimension of the coset does not include the $U(1)$ factor, so in the notation of (6.2) we have

$$M = \mathrm{dim(coset)} + 1. \tag{6.9}$$

By comparing the coefficients of m^2 in (6.1) and (6.8) we indeed reproduce the sum rule (6.2).

7 Conclusions

Starting from a chiral random matrix theory with the symmetries of the Dirac operator in an arbitrary background field, we have derived the finite volume partition functions

that were used as starting point in the derivation of the Leutwyler-Smilga sum rules in [6, 11]. A derivation of the simplest Leutwyler-Smilga sum rules shows in a natural way that they depend on the number of Goldstone bosons per flavor.

Our results explain the miracle that effective Lagrangians and chiral random matrix theory produce the same sum rules. The advantage of chiral random matrix theory is that it leads naturally to three classes of sum rules. The absence of a third class in [6] motivated the studies in [10] with the result that QCD with two colors in the fundamental representation constitutes a separate universality class. In the present work we have shown that then the Dirac operator is real and that the corresponding random matrix theory (chGOE) is equivalent to an effective theory based on the coset $SU(2N_f)/Sp(2N_f)$. For three or more colors in the fundamental representation the Dirac operator is complex and the corresponding random matrix theory with complex matrix elements (chGUE) is equivalent to a finite volume partition function based on the coset $SU_R(N_f) \times SU_L(N_f)/SU(N_f)$. The last universality class is for a Dirac operator with adjoint fermions and two or more colors. Then the Dirac operator is quaternion real, and in this paper we have shown that the corresponding random matrix theory with quaternion real matrix elements is equivalent to a finite volume partition function based on the coset $SU(N_f)/O(N_f)$.

Apart from two external parameters, the vacuum angle and the mass matrix, the random matrix theory and the corresponding finite volume partition function depend on only one dynamical parameter of QCD: the chiral condensate. The finite volume partition function, and thus the chiral random matrix theory, provides us with the mass dependence of the QCD partition function. However, this does not imply that all properties of the chiral random matrix model are physical. For example, the average level density, which has a semicircular shape, is certainly unphysical. This leads to the question which properties of the chiral random matrix theory actually determine the finite volume partition function.

In the finite volume partition function the mass occurs in the combination mV, whereas in the QCD partition function or in the random matrix theory, the mass occurs as m/λ, where λ is an eigenvalue of the Dirac operator. This suggests that the effective theory is only sensitive to the distribution of eigenvalues on a scale $1/V$. The spectral density on this scale, also called the microscopic spectral density, is well defined. From the work on quantum chaos and mesoscopic systems (including nuclei) [32] we know that the

distribution of eigenvalues on a microscopic scale is universal and is given by random matrix theory. The most impressive piece of work in the present context is by Slevin and Nagao [33] who studied the logarithm of the transfer matrix of a mesoscopic system in a magnetic field for the Hofstadter model. The random matrix theory of this system has the symmetries of the Dirac operator for $N_f = 0$ and three or more colors in the fundamental representation. Indeed, by numerical computations, they found that the microscopic spectral density is given by the corresponding chiral random matrix theory. This led us to the conjecture that the so called microscopic spectral density is universal. However, this does not imply that it can be determined from the finite volume partition function. For example, for one flavor, in each of the three cases, Goldstone bosons are absent and the finite volume partition function is the same but the random matrix theory is different. This shows that the low-lying spectrum of the Dirac operator cannot be derived from the complete set of Leutwyler-Smilga sum rules.

We conclude with the statement that although the finite volume partition function does not determine the microscopic spectral density, there is ample evidence that if combined with universality according to the anti-unitary symmetries of the Dirac operator this leads to a unique prediction of the spectral density of the Dirac operator on a scale of no more than a finite number of eigenvalues from zero.

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